

# LOW REGULARITY SOLUTIONS FOR THE (2+1) - DIMENSIONAL MAXWELL-KLEIN-GORDON EQUATIONS IN TEMPORAL GAUGE

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**ABSTRACT.** The Maxwell-Klein-Gordon equations in 2+1 dimensions in temporal gauge are locally well-posed for low regularity data even below energy level. The corresponding (3+1)-dimensional case was considered by Yuan. Fundamental for the proof is a partial null structure in the nonlinearity which allows to rely on bilinear estimates in wave-Sobolev spaces by d'Ancona, Foschi and Selberg, on an  $(L_x^p L_t^q)$  - estimate for the solution of the wave equation, and on the proof of a related result for the Yang-Mills equations by Tao.

## 1. INTRODUCTION AND MAIN RESULTS

Consider the Maxwell-Klein-Gordon equations

$$\partial^\alpha F_{\alpha\beta} = -Im(\phi \overline{D_\beta \phi}) \quad (1)$$

$$D^\mu D_\mu \phi = m^2 \phi \quad (2)$$

in Minkowski space  $\mathbb{R}^{1+2} = \mathbb{R}_t \times \mathbb{R}_x^2$  with metric  $diag(-1, 1, 1)$ . Greek indices run over  $\{0, 1, 2\}$ , Latin indices over  $\{1, 2\}$ , and the usual summation convention is used. Here  $m \in \mathbb{R}$  and

$$\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}, A_\alpha : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}, F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, D_\mu = \partial_\mu + iA_\mu.$$

$A_\mu$  are the gauge potentials,  $F_{\mu\nu}$  is the curvature. We use the notation  $\partial_\mu = \frac{\partial}{\partial x_\mu}$ , where we write  $(x^0, x^1, x^2) = (t, x^1, x^2)$  and also  $\partial_0 = \partial_t$ .

Setting  $\beta = 0$  in (1) we obtain the Gauss-law constraint

$$\partial^j F_{j0} = -Im(\phi \overline{D_0 \phi}). \quad (3)$$

The system (1),(2) is invariant under the gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi, \phi \rightarrow \phi' = e^{i\chi} \phi, D_\mu \rightarrow D'_\mu = \partial_\mu + iA'_\mu.$$

This allows to impose an additional gauge condition. We exclusively consider the temporal gauge

$$A_0 = 0. \quad (4)$$

In this gauge the system (1),(2) is equivalent to

$$\partial_t \partial^j A_j = Im(\phi \overline{\partial_t \phi}) \quad (5)$$

$$\square A_j = \partial_j(\partial^k A_k) - Im(\phi \overline{\partial_j \phi}) + A_j |\phi|^2 \quad (6)$$

$$\square \phi = -i(\partial^k A_k) \phi - 2iA^k \partial_k \phi + A^k A_k \phi + m^2 \phi, \quad (7)$$

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where  $\square = -\partial_t^2 + \Delta$  is the d'Alembert operator.

Other choices of the gauge are the Coulomb gauge  $\partial^j A_j = 0$  and the Lorenz gauge  $\partial^\mu A_\mu = 0$ .

Let us make some historical remarks. Most of the results were given in 3+1 dimensions. Klainerman and Machedon [KM] proved global well-posedness in energy space in Coulomb gauge and temporal gauge. Local well-posedness in Coulomb gauge for data for  $\phi$  in the Sobolev space  $H^s$  and for  $A$  in  $H^r$  with  $r = s > 1/2$ , i.e., almost down to the critical space with respect to scaling, was shown by Machedon and Sterbenz [MS]. In Lorenz gauge the global well-posedness result in energy space is due to Selberg and Teshafun [ST]. The author [P] proved local well-posedness for  $s = \frac{3}{4} + \epsilon$  and  $r = \frac{1}{2} + \epsilon$ . In temporal gauge Yuan [Y] obtained local well-posedness for  $s = r > \frac{3}{4}$  in  $X^{s,b}$ -spaces and global well-posedness for  $s = r = 1$ . The author [P1] proved that the finite energy solutions are also unique in the natural solution spaces. These results in temporal gauge rely on a similar result by Tao [T1] for the Yang-Mills equations and small data.

In 2+1 dimensions Moncrief [M] proved global well-posedness in Lorenz gauge for data in  $H^2$ . Local well-posedness in Lorenz gauge for  $s = \frac{3}{4} + \epsilon$  and  $r = \frac{1}{4} + \epsilon$  was shown by the author [P]. In Coulomb gauge local well-posedness for  $s = r = \frac{1}{2} + \epsilon$  and also for  $s = \frac{5}{8} + \epsilon$ ,  $r = \frac{1}{4} + \epsilon$  was obtained by Czubak and Pikula [CP].

In the present paper we exclusively consider the (2+1)-dimensional case in the temporal gauge and prove local well-posedness for data under minimal regularity assumptions. We need  $\phi(0) \in H^s$ ,  $(\partial_t \phi)(0) \in H^{s-1}$ ,  $A^{df}(0) \in H^r$ ,  $(\partial_t A^{df})(0) \in H^{r-1}$ ,  $|\nabla|^{\tilde{\epsilon}} A^{cf}(0) \in H^{l-\tilde{\epsilon}}$ , where  $A^{df}$  and  $A^{cf}$  denote the divergence-free and "curl-free" part of  $A$ , respectively, where an admissible choice is  $s = l = \frac{1}{2} + \frac{1}{14} +$ ,  $r = \frac{1}{4} +$ , and also  $s = r = l = \frac{1}{2} + \frac{1}{12} +$ . Uniqueness holds in spaces of Bourgain-Klainerman-Machedon type. If  $s = r = l = 1$  we even obtain unconditional uniqueness in the natural solution spaces. For a precise statement we refer to Theorem 1.1. We make use of a partial null structure of the nonlinearities and use bilinear estimates in wave-Sobolev spaces which were given systematically by d'Ancona, Foschi and Selberg [AFS]. We also need a powerful variant of Strichartz' estimates which gives an estimate for the  $L_x^6 L_t^2$ -norm of the solution of the wave equation which goes back to Tataru [KMBT]. The 3-dimensional variant was used by Tao [T1] for the more general Yang-Mills equation. Tao's hybrid estimates in this paper for the product of functions in wave-Sobolev spaces  $X_{|\tau|=|\xi|}^{s,b}$  and in product Sobolev spaces  $X_{\tau=0}^{s,b}$  (cf. the definition of the spaces below) are fundamental for our calculations.

We denote both the Fourier transform with respect to space and time and with respect to space by  $\hat{\cdot}$ . The operator  $|\nabla|^\alpha$  is defined by  $(\mathcal{F}(|\nabla|^\alpha f))(\xi) = |\xi|^\alpha (\mathcal{F}f)(\xi)$  and similarly  $\langle \nabla \rangle^\alpha$ , where  $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ . The inhomogeneous and homogeneous Sobolev spaces are denoted by  $H^{s,p}$  and  $\dot{H}^{s,p}$ , respectively. For  $p = 2$  we simply denote them by  $H^s$  and  $\dot{H}^s$ . We repeatedly use the Sobolev embeddings  $H^{s,p} \hookrightarrow L^q$  for  $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{s}{2}$ , and  $\dot{H}^{s,p} \hookrightarrow L^q$  for  $\frac{1}{q} = \frac{1}{p} - \frac{s}{2}$ , and  $1 < p \leq q < \infty$ . We also use the notation  $a \pm := a \pm \epsilon$  for a sufficiently small  $\epsilon > 0$ , so that  $a - - < a - < a < a + < a + +$ .

The standard space  $X_\pm^{s,b}$  of Bourgain-Klainerman-Machedon type (which was already considered by M. Beals [B]) belonging to the half waves is the completion of the Schwarz space  $\mathcal{S}(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{X_\pm^{s,b}} = \| \langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \hat{u}(\tau, \xi) \|_{L_{\tau\xi}^2}.$$

Similarly we define the wave-Sobolev space  $X_{|\tau|=|\xi|}^{s,b}$  with norm

$$\|u\|_{X_{|\tau|=|\xi|}^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widehat{u}(\tau, \xi)\|_{L_{\tau\xi}^2}$$

and also  $X_{\tau=0}^{s,b}$  with norm

$$\|u\|_{X_{\tau=0}^{s,b}} = \|\langle \xi \rangle^s \langle \tau \rangle^b \widehat{u}(\tau, \xi)\|_{L_{\tau\xi}^2}.$$

We also define  $X_{\pm}^{s,b}[0, T]$  as the space of the restrictions of functions in  $X_{\pm}^{s,b}$  to  $[0, T] \times \mathbb{R}^2$  and similarly  $X_{|\tau|=|\xi|}^{s,b}[0, T]$  and  $X_{\tau=0}^{s,b}[0, T]$ . We frequently use the estimate  $\|u\|_{X_{\pm}^{s,b}} \leq \|u\|_{X_{|\tau|=|\xi|}^{s,b}}$  for  $b \leq 0$  and the reverse estimate for  $b \geq 0$ .

We decompose  $A = (A_1, A_2)$  into its divergence-free part  $A^{df}$  and its "curl-free" part  $A^{cf}$ :

$$A = A^{df} + A^{cf},$$

where

$$\begin{aligned} A^{df} &:= PA := (-\Delta)^{-1}(\partial_2(\partial_1 A_2 - \partial_2 A_1), -\partial_1(\partial_1 A_2 - \partial_2 A_1)) \\ &= (R_2(R_1 A_2 - R_2 A_1), -R_1(R_1 A_2 - R_2 A_1)), \\ A^{cf} &:= -(-\Delta)^{-1} \nabla \operatorname{div} A = -(R_1(R_1 A_1 + R_2 A_2), R_2(R_1 A_1 + R_2 A_2)), \end{aligned}$$

and the Riesz-transform  $R_j$  is defined by  $R_j = |\nabla|^{-1} \partial_j$ .

Then we obtain the equivalent system

$$\partial_t A^{cf} = -(-\Delta)^{-1} \nabla \operatorname{Im}(\phi \overline{\partial_t \phi}) \quad (8)$$

$$\square A^{df} = -P(\operatorname{Im}(\phi \overline{\nabla \phi}) - A|\phi|^2) \quad (9)$$

$$\square \phi = -i(\partial^j A_j^{cf})\phi - 2iA_j^{df} \partial^j \phi - 2iA_j^{cf} \partial^j \phi + A^j A_j \phi + m^2 \phi. \quad (10)$$

Defining

$$\phi_{\pm} = \frac{1}{2}(\phi \pm i\langle \nabla \rangle^{-1} \partial_t \phi) \iff \phi = \phi_+ + \phi_-, \quad \partial_t \phi = i\langle \nabla \rangle(\phi_+ - \phi_-)$$

$$A_{\pm}^{df} = \frac{1}{2}(A^{df} \pm i\langle \nabla \rangle^{-1} \partial_t A^{df}) \iff A^{df} = A_+^{df} + A_-^{df}, \quad \partial_t A^{df} = i\langle \nabla \rangle(A_+^{df} - A_-^{df})$$

we can rewrite (8),(9),(10) as

$$\partial_t A^{cf} = -(-\Delta)^{-1} \nabla \operatorname{Im}(\phi \overline{\partial_t \phi}) \quad (11)$$

$$(-i\partial_t \pm \langle \nabla \rangle) A_{\pm}^{df} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S. \text{ of (9)} - A^{df}) \quad (12)$$

$$(-i\partial_t \pm \langle \nabla \rangle) \phi_{\pm} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S. \text{ of (10)} - \phi). \quad (13)$$

The initial data are transformed as follows:

$$\phi_{\pm}(0) = \frac{1}{2}(\phi(0) \pm i^{-1} \langle \nabla \rangle^{-1} (\partial_t \phi)(0)) \quad (14)$$

$$A_{\pm}^{df}(0) = \frac{1}{2}(A^{df}(0) \pm i^{-1} \langle \nabla \rangle^{-1} (\partial_t A^{df})(0)). \quad (15)$$

Our main result is preferably formulated in terms of the system (8),(9),(10).

**Theorem 1.1.** *1. Assume  $r > \frac{1}{4}$ ,  $l \geq s > \max(\frac{1}{2} + \frac{l}{8}, \frac{1}{4} + \frac{l}{2}, \frac{1}{4} + \frac{r}{2}, \frac{7}{16} + \frac{r}{4})$ ,  $r + \frac{1}{2} > s \geq r - \frac{1}{2}$ ,  $s > l - \frac{1}{2}$  and  $\tilde{\epsilon} > 0$  sufficiently small. Let  $\phi_0 \in H^s(\mathbb{R}^2)$ ,  $\phi_1 \in H^{s-1}(\mathbb{R}^2)$ ,  $a = a^{df} + a^{cf}$ ,  $a' = a'^{df} + a'^{cf}$  be given with  $a^{df} \in H^r(\mathbb{R}^2)$ ,  $|\nabla|^{\tilde{\epsilon}} a^{cf} \in H^{l-\tilde{\epsilon}}(\mathbb{R}^2)$ ,  $a'^{df} \in H^{r-1}(\mathbb{R}^2)$ , which satisfy the compatibility condition*

$$\partial^j a'_j = \operatorname{Im}(\phi_0 \overline{\phi_1}). \quad (16)$$

*Then there exists  $T > 0$ , such that (8),(9),(10) with initial conditions  $\phi(0) = \phi_0$ ,  $(\partial_t \phi)(0) = \phi_1$ ,  $A^{df}(0) = a^{df}$ ,  $(\partial_t A^{df})(0) = a'^{df}$ ,  $A^{cf}(0) = a^{cf}$  has a unique local solution*

$$\phi = \phi_+ + \phi_-, \quad A = A_+^{df} + A_-^{df} + A^{cf}$$

with

$$\phi_{\pm} \in X_{\pm}^{s, \frac{1}{2}+\epsilon}[0, T], \quad A_{\pm}^{df} \in X_{\pm}^{r, \frac{3}{4}+\epsilon}[0, T], \quad |\nabla|^{\bar{\epsilon}} A^{cf} \in X_{\tau=0}^{l-\bar{\epsilon}, \frac{1}{2}+\epsilon-}[0, T],$$

where  $\epsilon > 0$  is sufficiently small.

2. This solution satisfies

$$\phi_{\pm} \in C^0([0, T], H^s(\mathbb{R}^2)), \quad A_{\pm}^{df} \in C^0([0, T], H^r(\mathbb{R}^2)),$$

$$|\nabla|^{\bar{\epsilon}} A^{cf} \in C^0([0, T], H^{l-\bar{\epsilon}}(\mathbb{R}^2)).$$

In the case  $s = r = l = 1$  the solution is (unconditionally) unique in these spaces.

**Remarks:**

- The compatibility condition (16), which is necessary in view of (3), determines  $a'^{cf}$  as  $a'^{cf} = -(-\Delta)^{-1} \nabla(\operatorname{Im}(\phi_0 \bar{\phi}_1))$ . It is not difficult to see that  $a'^{cf}$  fulfills  $|\nabla|^{\bar{\epsilon}} a'^{cf} \in H^{l-1-\bar{\epsilon}}(\mathbb{R}^2)$ . One only has to show that

$$\| |\nabla|^{-1+\bar{\epsilon}}(\phi_0 \bar{\phi}_1) \|_{H^{l-1-\bar{\epsilon}}} \lesssim \|\phi_0\|_{H^s} \|\phi_1\|_{H^{s-1}}.$$

By duality this is equivalent to

$$\|\phi_0 \phi_2\|_{H^{1-s}} \lesssim \|\phi_0\|_{H^s} \| |\nabla|^{1-\bar{\epsilon}} \phi_2 \|_{H^{1-l+\bar{\epsilon}}}.$$

In the case of high frequencies of  $\phi_2$  this follows from the Sobolev multiplication law (17) using  $2s - l > 0$ , and the low frequency case can be easily handled using  $s > \frac{1}{2}$ .

- The minimal regularity assumptions are given by  $r = \frac{1}{4} +$ ,  $l = s = \frac{1}{2} + \frac{1}{14} +$ .
- If one wants to have the same regularity for  $\phi$  and  $A$  one also checks that  $r = l = s = \frac{1}{2} + \frac{1}{12} +$  is admissible.
- The choice  $r = l = s = 1$  is of course admissible.

Fundamental for us are the following estimates. We frequently use the classical Sobolev multiplication law in dimension two :

$$\|uv\|_{H^{-s_0}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}, \quad (17)$$

if  $s_0 + s_1 + s_2 \geq 1$  and  $s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2)$ , where at most one of these inequalities is an equality.

The corresponding bilinear estimates in wave-Sobolev spaces were proven by d'Ancona, Foschi and Selberg in the two-dimensional case in [AFS] in a form which includes some more limit cases which we do not need.

**Proposition 1.1.** For  $s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R}$  and  $u, v \in \mathcal{S}(\mathbb{R}^{2+1})$  the estimate

$$\|uv\|_{X_{|\tau|=|\xi|}^{-s_0, -b_0}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{s_1, b_1}} \|v\|_{X_{|\tau|=|\xi|}^{s_2, b_2}}$$

holds, provided the following conditions are satisfied:

$$b_0 + b_1 + b_2 > \frac{1}{2}, \quad b_0 + b_1 \geq 0, \quad b_0 + b_2 \geq 0, \quad b_1 + b_2 \geq 0$$

$$\begin{aligned}
s_0 + s_1 + s_2 &> \frac{3}{2} - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 &> 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2) \\
s_0 + s_1 + s_2 &> \frac{1}{2} - \min(b_0, b_1, b_2) \\
s_0 + s_1 + s_2 &> \frac{3}{4} \\
(s_0 + b_0) + 2s_1 + 2s_2 &> 1 \\
2s_0 + (s_1 + b_1) + 2s_2 &> 1 \\
2s_0 + 2s_1 + (s_2 + b_2) &> 1
\end{aligned}$$

$$s_1 + s_2 \geq \max(0, -b_0), \quad s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).$$

Moreover we need the standard Strichartz estimate combined with the transfer principle (for a proof see [S1], Theorem 8):

$$\|u\|_{L_{xt}^6} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}, \frac{1}{2}+}} \quad (18)$$

and the following estimate, which essentially goes back to Tataru [KMBT].

**Lemma 1.1.** *For  $2 \leq p \leq 6$  the following estimates hold:*

$$\begin{aligned}
\|u\|_{L_x^p L_t^2} &\lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p}), \frac{3}{2}(\frac{1}{2}-\frac{1}{p})+}}, \\
\|u\|_{L_x^p L_t^{2+}} &\lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})+, \frac{3}{2}(\frac{1}{2}-\frac{1}{p})+}}.
\end{aligned}$$

*Proof.* By [KMBT], Thm. B2 we obtain  $\|\mathcal{F}_t u\|_{L_x^2 L_\tau^6} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{6}}}$ , if  $u = e^{it|\nabla|} u_0$  and  $\mathcal{F}_t$  denotes the Fourier transform with respect to time. This implies by Plancherel and Minkowski's inequality

$$\|u\|_{L_x^6 L_t^2} = \|\mathcal{F}_t u\|_{L_x^6 L_\tau^2} \leq \|\mathcal{F}_t u\|_{L_\tau^2 L_x^6} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{6}}}.$$

The transfer principle [S1], Prop. 8 implies

$$\|u\|_{L_x^6 L_t^2} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{6}, \frac{1}{2}+}}. \quad (19)$$

Interpolation with (18) gives

$$\|u\|_{L_x^6 L_t^{2+}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{\frac{1}{6}+, \frac{1}{2}+}}. \quad (20)$$

Interpolation of the last two inequalities with the trivial identity  $\|u\|_{L_x^2 L_t^2} = \|u\|_{X_{|\tau|=|\xi|}^{0,0}}$  completes the proof.  $\square$

## 2. PROOF OF THE THEOREM

We now consider the Cauchy problem (11), (12), (13), (14), (15). Klainerman and Machedon detected that  $A^{df} \cdot \nabla \phi$  and  $P(\phi \overline{\nabla \phi})_k$  are null forms. An elementary calculation namely shows that

$$A_i^{df} \partial^i \phi = Q_{12}(\phi, |\nabla|^{-1}(R_1 A_2 - R_2 A_1)) \quad (21)$$

and

$$P(\phi \overline{\nabla \phi})_1 = -2i R_2 |\nabla|^{-1} Q_{12}(Re \phi, Im \phi) \quad (22)$$

$$P(\phi \overline{\nabla \phi})_2 = 2i R_1 |\nabla|^{-1} Q_{12}(Re \phi, Im \phi), \quad (23)$$

where the null form  $Q_{12}$  is defined by

$$Q_{12}(u, v) := \partial_1 u \partial_2 v - \partial_1 u \partial_2 v.$$

In order to estimate these null forms we also use the following estimate for the angle  $\angle(\xi_1, \xi_2)$  between two vectors  $\xi_1$  and  $\xi_2$ .

**Lemma 2.1.** *Assume  $0 \leq \alpha, \beta, \gamma \leq \frac{1}{2}$  and  $\xi_i \in \mathbb{R}^2$ ,  $\tau_i \in \mathbb{R}$  ( $i = 1, 2, 3$ ) with  $\xi_1 + \xi_2 + \xi_3 = 0$ ,  $\tau_1 + \tau_2 + \tau_3 = 0$ . Then the following estimate holds for independent signs  $\pm$  and  $\pm'$ :*

$$\angle(\pm\xi_1, \pm'\xi_2) \lesssim \left( \frac{\langle -\tau_1 \pm |\xi_1| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\alpha + \left( \frac{\langle -\tau_2 \pm' |\xi_2| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\beta + \left( \frac{\langle |\tau_3| - |\xi_3| \rangle}{\min(\langle \xi_1 \rangle, \langle \xi_2 \rangle)} \right)^\gamma. \quad (24)$$

For a proof see for example [S], Lemma 2.1.

**Proof of Theorem 1.1. Proof of part 1:** We use (21), (22), (23). By a contraction argument the local existence and uniqueness proof is reduced to suitable multilinear estimates for the right hand sides of (11), (12), (13). For (12), e.g., we make use of the following well-known estimate for a solution of the linear equation  $(-i\partial_t \pm \langle \nabla \rangle) A_\pm^{df} = G$ , namely

$$\|A_\pm^{df}\|_{X_\pm^{k,b}[0,T]} \lesssim \|A_\pm^{df}(0)\|_{H^k} + T^{b'-b} \|G\|_{X_\pm^{k,b'-1}[0,T]},$$

which holds for  $k \in \mathbb{R}$ ,  $\frac{1}{2} < b \leq b' < 1$  and  $0 < T \leq 1$ .

Thus the local existence and uniqueness for large data (in which case we have to choose  $b < b'$ ), in the regularity class

$$\phi_\pm \in X_\pm^{s, \frac{1}{2}+\epsilon}[0, T], \quad A_\pm^{df} \in X_\pm^{r, \frac{3}{4}+\epsilon}[0, T], \quad |\nabla|^{\tilde{\epsilon}} A^{cf} \in X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}[0, T]$$

can be reduced to the following estimates for independent signs  $\pm$ ,  $\pm'$ ,  $\pm''$ :

$$\| |\nabla|^{-1+\tilde{\epsilon}} (\phi_1 \partial_t \phi_2) \|_{X_{\tau=0}^{l-\tilde{\epsilon}, -\frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (25)$$

$$\| |\nabla|^{-1} Q_{ij}(\phi_1, \phi_2) \|_{X_{\pm''}^{r-1, -\frac{1}{4}+2\epsilon}} \lesssim \|\phi_1\|_{X_\pm^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\pm'}^{s, \frac{1}{2}+\epsilon}}, \quad (26)$$

$$\| Q_{ij}(|\nabla|^{-1} \phi_1, \phi_2) \|_{X_{\pm''}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \|\phi_1\|_{X_\pm^{r, \frac{3}{4}+\epsilon}} \|\phi_2\|_{X_{\pm'}^{s, \frac{1}{2}+\epsilon}}, \quad (27)$$

$$\| \nabla A \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} + \| A \nabla \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \| |\nabla|^{\tilde{\epsilon}} A \|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (28)$$

$$\| A \phi_1 \phi_2 \|_{X_{|\tau|=|\xi|}^{r-1, -\frac{1}{4}+2\epsilon}} \lesssim \min(\|A\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}, \| |\nabla|^{\tilde{\epsilon}} A \|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}}) \prod_{i=1}^2 \|\phi_i\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (29)$$

$$\| A_1 A_2 \phi \|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \prod_{i=1}^2 \min(\|A_i\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}, \| |\nabla|^{\tilde{\epsilon}} A_i \|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}}) \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \quad (30)$$

**Proof of (27):** The Fourier multiplier of  $Q_{12}(|\nabla|^{-1} \phi_1, \phi_2)$  is bounded by

$$\frac{|\xi_1 \times \xi_2|}{|\xi_1|} \lesssim |\xi_2| \angle(\pm\xi_1, \pm'\xi_2), \quad (31)$$

where  $\xi_1 \times \xi_2 := \xi_{11}\xi_{22} - \xi_{21}\xi_{12}$ . If  $\xi_1 + \xi_2 + \xi_3 = 0$  we also have

$$\frac{|\xi_1 \times \xi_2|}{|\xi_1|} = \frac{|\xi_1 \times \xi_3|}{|\xi_1|} \lesssim |\xi_3| \angle(\pm\xi_1, \pm''\xi_3). \quad (32)$$

1. In the case  $|\xi_3| \gtrsim \max(|\xi_1|, |\xi_2|)$  we use (31). It suffices to show

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^s \langle -\tau_2 \pm' |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-s} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} \cdot \angle(\pm\xi_1, \pm'\xi_2) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (33)$$

The Fourier transforms are nonnegative without loss of generality. Here  $*$  denotes integration over  $\sum_{i=1}^3 \xi_i = 0$ ,  $\sum_{i=1}^3 \tau_i = 0$  and  $d\xi d\tau = d\xi_1 d\xi_2 d\xi_3 d\tau_1 d\tau_2 d\tau_3$ .

We use (24) with  $\alpha = \beta = \frac{1}{2}$ ,  $\gamma = \frac{1}{2}-$ .

**1.1.**  $|\xi_1| \leq |\xi_2|$ . If the first term on the r.h.s. of (24) is dominant we use  $|\xi_3| \sim |\xi_2|$  and reduce to

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

which follows from Prop. 1.1 for  $r > \frac{1}{4}$ , where we need the factor  $\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{4}+}$  in the denominator. For the second and third term on the r.h.s. of (24) we only have to show

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \widehat{u}_2(\xi_2, \tau_2) \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

and

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{r+\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

respectively, both of which follow from Prop. 1.1 for  $r > \frac{1}{4}$ .

**1.2.**  $|\xi_1| \geq |\xi_2|$ . Using  $|\xi_3| \sim |\xi_1|$  the l.h.s. of (33) is bounded by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau$$

for the first term on the r.h.s. of (24). Similarly for the second and third term we obtain the bounds

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau$$

and

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-s+r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau,$$

respectively, all of which are bounded by  $\prod_{i=1}^3 \|u_i\|_{L_{xt}^2}$  for  $r > \frac{1}{4}$ ,  $s > \frac{1}{2}$  and  $s \leq r+1$  by Prop. 1.1.

**2.** In the case  $|\xi_3| \ll |\xi_1| \sim |\xi_2|$  we use (32). It suffices to show

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3) |\xi_3|}{\langle \xi_3 \rangle^{1-s} \langle -\tau_3 \pm'' |\xi_3| \rangle^{\frac{1}{2}-}} \\ & \cdot \angle(\pm \xi_1, \pm'' \xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (34) \end{aligned}$$

Using  $|\xi_2| \gtrsim |\xi_3|$  and (24) with  $\alpha = \beta = \frac{1}{2}$ ,  $\gamma = \frac{1}{2}-$  and  $\xi_2$  permuted with  $\xi_3$  we bound the l.h.s. of (34) by

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \widehat{u}_2(\xi_2, \tau_2) \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau \\ & + \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{\frac{3}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \\ & + \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^r \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{4}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}-}} d\xi d\tau, \end{aligned}$$

which gives (34) by Prop. 1.1 and completes the proof of (27).

**Proof of (26):** We recall (31) and (32) and obtain the following bounds for the Fourier multiplier of  $Q_{12}(\phi_1, \phi_2)$  :

$$|\xi_1 \times \xi_2| \lesssim |\xi_1| |\xi_2| \angle(\pm \xi_1, \pm' \xi_2), \quad (35)$$

$$|\xi_1 \times \xi_2| \lesssim |\xi_1| |\xi_3| \angle(\pm \xi_1, \pm'' \xi_3). \quad (36)$$

1. In the case  $|\xi_3| \gtrsim \max(|\xi_1|, |\xi_2|)$  we use (35) and reduce the desired estimate to

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1) |\xi_1|}{\langle \xi_1 \rangle^s \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2) |\xi_2|}{\langle \xi_2 \rangle^s \langle -\tau_2 \pm' |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{|\xi_3| \langle \xi_3 \rangle^{1-r} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} \cdot \angle(\pm \xi_1, \pm' \xi_2) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (37)$$

By symmetry we may assume  $|\xi_1| \leq |\xi_2|$ . We estimate  $\angle(\pm \xi_1, \pm' \xi_2)$  by (24) with  $\alpha = \beta = \frac{1}{2}$ ,  $\gamma = \frac{1}{4}-$ . We estimate the l.h.s. of (37) concerning the first term on the r.h.s. of (24) using  $|\xi_3| \sim |\xi_2|$  by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-r+1} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} d\xi d\tau,$$

which gives (37) by Prop. 1.1, where we used  $s > \frac{1}{4} + \frac{r}{2}$  and  $s \geq r - \frac{1}{2}$ . For the second term on the r.h.s. of (24) we control the l.h.s. of (37) by

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s+1-r} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{2s+\frac{1}{2}-r} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \widehat{u}_2(\xi_2, \tau_2) \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} d\xi d\tau. \end{aligned}$$

We apply Prop. 1.1 using  $s > \frac{1}{4} + \frac{r}{2}$  and  $s \geq r - 1$  to obtain (37). For the last term on the r.h.s. of (24) we estimate the l.h.s. of (37) using  $|\xi_3| \sim |\xi_2| \gtrsim |\xi_1|$  and  $s \geq r - 1$  as follows:

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{3}{4}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-r}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{2s-r+\frac{1}{4}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \end{aligned}$$

The last estimate follows from Prop. 1.1 using  $s > \frac{1}{4} + \frac{r}{2}$  again.

2. In the case  $|\xi_3| \ll |\xi_1| \sim |\xi_2|$  we use (36) and reduce the desired estimate to

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1) |\xi_1|}{\langle \xi_1 \rangle^s \langle -\tau_1 \pm |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-r} \langle -\tau_3 \pm'' |\xi_3| \rangle^{\frac{1}{4}-}} \cdot \angle(\pm \xi_1, \pm'' \xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \quad (38)$$

We estimate  $\angle(\pm \xi_1, \pm'' \xi_3)$  by (24) with  $\alpha = \beta = \frac{1}{2}$ ,  $\gamma = \frac{1}{4}-$ . We bound the l.h.s. of (38) for the first term on the r.h.s. of (24) (and similarly for the second term) using  $|\xi_1| \sim |\xi_2|$  and  $s \geq \frac{1}{2}$  by

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{3}{2}-r} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} d\xi d\tau \\ & \lesssim \int_* \widehat{u}_1(\xi_1, \tau_1) \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{2s-r+\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{4}-}} d\xi d\tau, \end{aligned}$$



which again implies (38) by Prop. 1.1 using  $s > \frac{1}{4} + \frac{r}{2}$ .

For the last term on the r.h.s. of (24) we estimate the l.h.s. of (38) by

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{1}{2}} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{1-r+\frac{1}{4}-}} d\xi d\tau.$$

If  $r < \frac{5}{4}$  we apply Prop. 1.1 using  $s > \frac{1}{4} + \frac{r}{2}$ ,  $s > r - \frac{1}{2}$ ,  $s > \frac{1}{2}$  and  $s > \frac{7}{16} + \frac{r}{4}$ , which implies (38).

If  $r \geq \frac{5}{4}$  we use  $|\xi_3| \ll |\xi_1| \sim |\xi_2|$  and obtain the bound

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{s-\frac{r}{2}+\frac{1}{8}-} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{s-\frac{r}{2}+\frac{1}{8}-} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\xi_3, \tau_3) d\xi d\tau,$$

which again implies (38) by Prop. 1.1 using  $s > \frac{1}{4} + \frac{r}{2}$ , completing the proof of (26).

**Proof of (25):** We first remark that the singularity of  $|\nabla|^{-1+\tilde{\epsilon}}$  ( $\tilde{\epsilon} > 0$ ) is harmless in two dimensions ([T], Cor. 8.2) and it can be replaced by  $\langle \nabla \rangle^{-1+\tilde{\epsilon}}$ . As a first step we use Sobolev's multiplication law (17) and obtain

$$|\int \int u_1 u_2 u_3 dx dt| \lesssim \|u_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|u_2\|_{X_{\tau=0}^{s, -\frac{1}{2}+\epsilon}} \|u_3\|_{X_{\tau=0}^{1-l, \frac{1}{2}-\epsilon}}$$

under the assumptions  $s > \frac{l}{2}$  and  $s > l-1$ . This implies taking the time derivative into account

$$\|\langle \nabla \rangle^{-1+\tilde{\epsilon}}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{l-\tilde{\epsilon}, -\frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}}. \quad (39)$$

In a second step we want to prove

$$\begin{aligned} \|\langle \nabla \rangle^{-1+\tilde{\epsilon}}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{l-\tilde{\epsilon}, -\frac{1}{2}+\epsilon}} + \|\langle \nabla \rangle^{-1+\tilde{\epsilon}}(\phi_2 \partial_t \phi_1)\|_{X_{\tau=0}^{l-\tilde{\epsilon}, -\frac{1}{2}+\epsilon}} \\ \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}}. \end{aligned} \quad (40)$$

If  $\widehat{\phi}_1(\xi_3, \tau_3)$  is supported in  $||\tau_3| - |\xi_3|| \gtrsim |\xi_3|$  we have the trivial bound

$$\|\phi_1\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \lesssim \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \quad (41)$$

so that (40) follows from (39). Assuming from now on  $||\tau_3| - |\xi_3|| \ll |\xi_3|$  we have to prove

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2} \quad (42)$$

where

$$m = \frac{(|\tau_2| + |\tau_3|) \chi_{||\tau_3| - |\xi_3|| \ll |\xi_3|}}{\langle \xi_1 \rangle^{1-l} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^s \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}}.$$

Since  $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$  and  $\tau_1 + \tau_2 + \tau_3 = 0$  we have

$$|\tau_2| + |\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}-\epsilon}. \quad (43)$$

Thus (42) is a consequence of the following three estimates:

$$\begin{aligned} |\int \int uvw dx dt| &\lesssim \|u\|_{X_{\tau=0}^{1-l, 0}} \|v\|_{X_{\tau=0}^{s, 0}} \|w\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \\ |\int \int uvw dx dt| &\lesssim \|u\|_{X_{\tau=0}^{1-l, 0}} \|v\|_{X_{\tau=0}^{s, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}} \\ |\int \int uvw dx dt| &\lesssim \|u\|_{X_{\tau=0}^{1-l, \frac{1}{2}-\epsilon}} \|v\|_{X_{\tau=0}^{s, 0}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon}}, \end{aligned}$$

which easily follow from Sobolev's multiplication law (17) using  $s > \frac{1}{4} + \frac{l}{2}$  and  $s > l - \frac{1}{2}$ .

We now come to the proof of (25) and remark that we may assume now that both functions  $\phi_1$  and  $\phi_2$  are supported in  $||\tau| - |\xi|| \ll |\xi|$ , because otherwise (25) is an immediate consequence of (40) and (41). Thus (25) follows if we can prove the following estimate:

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

where

$$m = \frac{|\tau_3| |\chi| |\tau_2| - |\xi_2| \ll |\xi_2| |\chi| |\tau_3| - |\xi_3| \ll |\xi_3|}{\langle \xi_1 \rangle^{1-l} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}}.$$

Since  $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$ ,  $\langle \tau_2 \rangle \sim \langle \xi_2 \rangle$  and  $\tau_1 + \tau_2 + \tau_3 = 0$  we obtain

$$|\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} + \langle \xi_2 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon}.$$

The first term is taken care of by the estimate

$$|\int \int uvw dx dt| \lesssim \|u\|_{X_{\tau=0}^{1-l,0}} \|v\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}},$$

which follows from Prop. 1.1 under the assumptions  $s > \frac{1}{8} + \frac{l}{2}$ ,  $s > \frac{1}{4} + \frac{l}{4}$ ,  $s > \frac{1}{2}$  and  $s > l - \frac{1}{2}$ .

In order to treat the second term on the right hand side we have to show

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \langle \xi_1 \rangle^{l-1} \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^{s-\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{s-\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}, \quad (44)$$

which is equivalent to

$$|\int \int uvw dx dt| \lesssim \|u\|_{X_{\tau=0}^{1-l, \frac{1}{2}-\epsilon}} \|v\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon}}.$$

We consider first the case  $l \leq 1$ . By Hölder's inequality we obtain

$$|\int \int uvw dx dt| \leq \|u\|_{L_x^q L_t^r} \|v\|_{L_x^p L_t^z} \|w\|_{L_x^p L_t^z}$$

where we choose  $\frac{1}{q} = \frac{l}{2}$ ,  $\frac{1}{p} = \frac{1}{2} - \frac{l}{4}$ ,  $\frac{1}{r} = \epsilon$ ,  $\frac{1}{z} = \frac{1}{2} - \frac{\epsilon}{2}$ , so that we obtain by Sobolev

$$\|u\|_{L_x^q L_t^r} \lesssim \|u\|_{H_x^{1-l} H_t^{\frac{1}{2}-\epsilon}} \lesssim \|u\|_{X_{\tau=0}^{1-l, \frac{1}{2}-\epsilon}}.$$

Because  $z = 2+$  and  $2 \leq p \leq 6$  (for  $l \leq \frac{4}{3}$ ) we may apply Lemma 1.1 and obtain the desired estimate for  $v$  and also  $w$ :

$$\|v\|_{L_x^p L_t^z} \lesssim \|v\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}+}} \leq \|v\|_{X_{|\tau|=|\xi|}^{s-\frac{1}{2}-\epsilon, \frac{1}{2}+}},$$

provided  $\frac{1}{2}(\frac{1}{2}-\frac{1}{p}) < s - \frac{1}{2} \iff s > \frac{1}{2} + \frac{l}{8}$ . Here the decisive lower bound for  $s$  is required, namely  $l = s > \frac{1}{2} + \frac{1}{14}$ , because we shall see below that we need  $l \geq s$  for the estimate (28). The proof of (44) in the case  $l \leq 1$  is complete.

Next we consider the case  $l > 1$ . The left hand side of (44) is bounded by

$$\begin{aligned} & \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^{s-l+\frac{1}{2}} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{s-\frac{1}{2}} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{2s-l} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau \\ & \lesssim \int_* \frac{\widehat{u}_1(\xi_1, \tau_1) \widehat{u}_2(\xi_2, \tau_2) \widehat{u}_3(\xi_3, \tau_3)}{\langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+} \langle |\tau_3| - |\xi_3| \rangle^{\frac{1}{2}+\epsilon}} d\xi d\tau, \end{aligned} \quad (45)$$

where we assumed w.l.o.g.  $|\xi_2| \geq |\xi_3|$ , so that  $\langle \xi_1 \rangle \lesssim \langle \xi_2 \rangle$ , and our assumptions  $s - l + \frac{1}{2} \geq 0$  and  $2s - l > \frac{1}{2}$ . By Sobolev and Lemma 1.1 we obtain

$$\|w\|_{L_x^\infty L_t^2} \lesssim \|w\|_{H_x^{\frac{1}{3}+6} L_t^2} \lesssim \|w\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+}}$$

which implies

$$\begin{aligned} \left| \int uvw \, dx dt \right| &\lesssim \|u\|_{L_x^2 L_t^2} \|v\|_{L_x^2 L_t^\infty} \|w\|_{L_x^\infty L_t^2} \\ &\lesssim \|u\|_{X_{\tau=0}^{0, \frac{1}{2}-\epsilon}} \|v\|_{X_{|\tau|=|\xi|}^{0, \frac{1}{2}+\epsilon}} \|w\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+}}. \end{aligned}$$

Thus (45) is bounded by  $\prod_{i=1}^3 \|u_i\|_{L_{xt}^2}$ , which completes the proof of (44) and also (25).

**Proof of (28):** This proof is similar to a related estimate for the Yang-Mills equation given by Tao [T1]. We have to show

$$\int_* m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

where

$$m = \frac{(|\xi_2| + |\xi_3|) \langle \xi_1 \rangle^{s-1}}{\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}-2\epsilon} \langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+\epsilon} |\xi_3|^\epsilon \langle \xi_3 \rangle^{l-\bar{\epsilon}} \langle \tau_3 \rangle^{\frac{1}{2}+\epsilon-}}.$$

Case 1:  $|\xi_2| \lesssim |\xi_1|$  ( $\Rightarrow |\xi_2| + |\xi_3| \lesssim |\xi_1|$ ).

We ignore the factor  $\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}-2\epsilon}$  and use the averaging principle ([T], Prop. 5.1) to replace  $m$  by

$$m' = \frac{\langle \xi_1 \rangle^s \chi_{|\tau_2| - |\xi_2| \sim 1} \chi_{|\tau_3| \sim 1}}{\langle \xi_2 \rangle^s |\xi_3|^\epsilon \langle \xi_3 \rangle^{l-\bar{\epsilon}}}.$$

Let now  $\tau_2$  be restricted to the region  $\tau_2 = T + O(1)$  for some integer  $T$ . Then  $\tau_1$  is restricted to  $\tau_1 = -T + O(1)$ , because  $\tau_1 + \tau_2 + \tau_3 = 0$ , and  $\xi_2$  is restricted to  $|\xi_2| = |T| + O(1)$ . The  $\tau_1$ -regions are essentially disjoint for  $T \in \mathbb{Z}$  and similarly the  $\tau_2$ -regions. Thus by Schur's test ([T], Lemma 3.11) we only have to show

$$\begin{aligned} \sup_{T \in \mathbb{Z}} \int_* \frac{\langle \xi_1 \rangle^s \chi_{\tau_1 = -T+O(1)} \chi_{\tau_2 = T+O(1)} \chi_{|\tau_3| \sim 1} \chi_{|\xi_2| = |T|+O(1)}}{\langle \xi_2 \rangle^s |\xi_3|^\epsilon \langle \xi_3 \rangle^{l-\bar{\epsilon}}} \prod_{i=1}^3 \widehat{u}_i(\xi_i, \tau_i) d\xi d\tau \\ \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}. \end{aligned}$$

The  $\tau$ -behaviour of the integral is now trivial, thus we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\langle \xi_1 \rangle^s \chi_{|\xi_2| = T+O(1)}}{\langle T \rangle^s |\xi_3|^\epsilon \langle \xi_3 \rangle^{l-\bar{\epsilon}}} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}. \quad (46)$$

Assuming now  $|\xi_3| \leq |\xi_1|$  (the other case being simpler) it only remains to consider the following two cases:

Case 1.1:  $|\xi_1| \sim |\xi_3| \gtrsim T$ . We now use our assumption  $l \geq s$ , so that it suffices to show

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i = 0} \frac{\chi_{|\xi_2| = T+O(1)}}{T^l} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

The l.h.s. is bounded by

$$\begin{aligned} & \sup_{T \in \mathbb{N}} \frac{1}{T^l} \|f_1\|_{L^2} \|f_3\|_{L^2} \|\mathcal{F}^{-1}(\chi_{|\xi|=T+O(1)} \widehat{f_2})\|_{L^\infty(\mathbb{R}^2)} \\ & \lesssim \sup_{T \in \mathbb{N}} \frac{1}{T^l} \|f_1\|_{L^2} \|f_3\|_{L^2} \|\chi_{|\xi|=T+O(1)} \widehat{f_2}\|_{L^1(\mathbb{R}^2)} \\ & \lesssim \sup_{T \in \mathbb{N}} \frac{T^{\frac{1}{2}}}{T^l} \prod_{i=1}^3 \|f_i\|_{L^2} \lesssim \prod_{i=1}^3 \|f_i\|_{L^2}, \end{aligned}$$

because one easily calculates that  $l > \frac{1}{2}$  under our assumptions.

Case 1.2:  $|\xi_1| \sim T \gtrsim |\xi_3|$ . In this case it suffices to show

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i=0} \frac{\chi_{|\xi_2|=T+O(1)}}{|\xi_3|^{\bar{\epsilon}} \langle \xi_3 \rangle^{l-\bar{\epsilon}}} \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \widehat{f_3}(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

Case 1.2.1:  $|\xi_3| \geq 1$ . An elementary calculation shows that the l.h.s. is bounded by

$$\sup_{T \in \mathbb{N}} \|\chi_{|\xi|=T+O(1)} * \langle \xi \rangle^{-2l} \|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \prod_{i=1}^3 \|f_i\|_{L_x^2} \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2},$$

using as in case 1.1 that  $l > \frac{1}{2}$ .

Case 1.2.2:  $|\xi_3| \leq 1$ . The l.h.s. is crudely estimated by

$$\begin{aligned} & \sup_{T \in \mathbb{N}} \|\chi_{|\xi|=T+O(1)} * \chi_{|\xi| \leq 1} |\xi|^{-2\bar{\epsilon}} \|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}} \prod_{i=1}^3 \|f_i\|_{L_x^2} \\ & \lesssim \left( \int_{|\xi| \leq 1} |\xi|^{-2\bar{\epsilon}} d\xi \right)^{\frac{1}{2}} \prod_{i=1}^3 \|f_i\|_{L_x^2} \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}. \end{aligned}$$

Case 2.  $|\xi_1| \ll |\xi_2|$  ( $\Rightarrow |\xi_2| + |\xi_3| \lesssim |\xi_2|$ ).

Exactly as in case 1 we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i=0} \frac{\langle T \rangle^{1-s} \chi_{|\xi_2|=T+O(1)}}{\langle \xi_1 \rangle^{1-s} |\xi_3|^{\bar{\epsilon}} \langle \xi_3 \rangle^{l-\bar{\epsilon}}} \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \widehat{f_3}(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2}.$$

Using  $|\xi_3| \sim |\xi_2| \sim T \gg |\xi_1|$  and  $l > \frac{1}{2}$  we crudely estimate:

$$\frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} |\xi_3|^{\bar{\epsilon}} \langle \xi_3 \rangle^{l-\bar{\epsilon}}} \sim \frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} T^l} \lesssim \frac{T^{1-s}}{\langle \xi_1 \rangle^{1-s} \langle \xi_1 \rangle^{s-\frac{1}{2}+T^{l-s+\frac{1}{2}-}}} \lesssim \frac{1}{\langle \xi_1 \rangle^{\frac{1}{2}+}}.$$

Thus we reduce to

$$\sup_{T \in \mathbb{N}} \int_{\sum_{i=1}^3 \xi_i=0} \frac{\chi_{|\xi_2|=T+O(1)}}{\langle \xi_1 \rangle^{\frac{1}{2}+}} \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \widehat{f_3}(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_x^2},$$

which can be shown as in Case 1.2. The proof of (28) is complete.

**Proof of (29):** Assume first that  $r \leq 1$ . We estimate by Sobolev's multiplication law (17) using  $s > \frac{1}{2}$ :

$$\begin{aligned} & \|A\phi_1\phi_2\|_{X_{|\tau|=|\xi|}^{r-1, -\frac{1}{2}+2\epsilon}} \lesssim \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} \lesssim \|A\|_{L_t^6 H_x^r} \|\phi_1\phi_2\|_{L_t^3 H_x^{0+}} \\ & \lesssim \|A\|_{L_t^6 H_x^r} \|\phi_1\|_{L_t^6 H_x^{\frac{1}{2}+}} \|\phi_2\|_{L_t^6 H_x^{\frac{1}{2}+}} \lesssim \|A\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \end{aligned}$$

Especially we have

$$\|A\phi_1\phi_2\|_{X_{|\tau|=|\xi|}^{0, -\frac{1}{2}+2\epsilon}} \lesssim \|A\|_{X_{|\tau|=|\xi|}^{1, \frac{3}{4}+\epsilon}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+, \frac{1}{2}+\epsilon}}.$$

The fractional Leibniz rule implies for  $r > 1$  :

$$\begin{aligned} \|A\phi_1\phi_2\|_{X_{|\tau|=|\xi|}^{r-1, -\frac{1}{2}+2\epsilon}} &\lesssim \|A\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{r-\frac{1}{2}+, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{r-\frac{1}{2}+, \frac{1}{2}+\epsilon}} \\ &\lesssim \|A\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \end{aligned}$$

by our assumption  $r - \frac{1}{2} < s$ .

Assume now again that  $r \leq 1$ . We obtain

$$\begin{aligned} \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} &\lesssim \|A\phi_1\phi_2\|_{L_t^2 L_x^p} \lesssim \|A\|_{L_t^6 L_x^4} \|\phi_1\|_{L_t^6 L_x^q} \|\phi_2\|_{L_t^6 L_x^q} \\ &\lesssim \|A\|_{L_t^6 \dot{H}_x^{\frac{1}{2}}} \|\phi_1\|_{L_t^6 H_x^{\frac{1}{4}+\frac{\epsilon}{2}}} \|\phi_2\|_{L_t^6 H_x^{\frac{1}{4}+\frac{\epsilon}{2}}} \lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{L_t^6 H_x^{l-\epsilon}} \|\phi_1\|_{L_t^6 H_x^s} \|\phi_2\|_{L_t^6 H_x^s} \\ &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \end{aligned}$$

where  $\frac{1}{p} = 1 - \frac{r}{2}$ ,  $\frac{1}{q} = \frac{3}{8} - \frac{r}{4}$ , so that by Sobolev we obtain  $L_x^p \hookrightarrow H_x^{r-1}$ ,  $H_x^{\frac{1}{4}+\frac{\epsilon}{2}} \hookrightarrow L_x^q$  and  $\dot{H}_x^{\frac{1}{2}} \hookrightarrow L_x^4$ . We used  $l \geq \frac{1}{2}$  and  $s \geq \frac{r}{2} + \frac{1}{4}$ . Especially we obtain for  $r = 1$  :

$$\|A\phi_1\phi_2\|_{L_t^2 L_x^2} \lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{L_t^6 H_x^{\frac{3}{4}}} \|\phi_2\|_{L_t^6 H_x^{\frac{3}{4}}}.$$

Next we consider the case  $r = \frac{3}{2}$ . By the fractional Leibniz rule we obtain

$$\begin{aligned} \|A\phi_1\phi_2\|_{L_t^2 H_x^{\frac{1}{2}}} &\lesssim \|A\|_{L_t^6 H_x^{\frac{1}{2}+, 2+}} \|\phi_1\|_{L_t^6 L_x^{\infty-}} \|\phi_2\|_{L_t^6 L_x^{\infty-}} + \|A\|_{L_t^6 L_x^{4+}} \|\phi_1\|_{L_t^6 H_x^{\frac{1}{2}, 4-}} \|\phi_2\|_{L_t^6 L_x^{\infty-}} \\ &\quad + \|A\|_{L_t^6 L_x^{4+}} \|\phi_1\|_{L_t^6 L_x^{\infty-}} \|\phi_2\|_{L_t^6 H_x^{\frac{1}{2}, 4-}} \\ &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{\frac{1}{2}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{L_t^6 H_x^{1-}} \|\phi_2\|_{L_t^6 H_x^{1-}} \\ &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{\frac{1}{2}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{1-, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{1-, \frac{1}{2}+\epsilon}}, \end{aligned}$$

This is enough, because  $s \geq 1$  and  $l > \frac{1}{2}$ .

By bilinear interpolation between the cases  $r = 1$  and  $r = \frac{3}{2}$  we easily obtain for  $1 < r < \frac{3}{2}$  :

$$\begin{aligned} \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{L_t^6 H_x^{\frac{1}{4}+\frac{\epsilon}{2}}} \|\phi_2\|_{L_t^6 H_x^{\frac{1}{4}+\frac{\epsilon}{2}}} \\ &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \end{aligned}$$

under our assumption  $s > \frac{r}{2} + \frac{1}{4}$ .

The remaining case  $r > \frac{3}{2}$  follows from the case  $r = \frac{3}{2}$  by the fractional Leibniz rule:

$$\begin{aligned} \|A\phi_1\phi_2\|_{L_t^2 H_x^{r-1}} &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{r-1, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{r-\frac{1}{2}, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{r-\frac{1}{2}, \frac{1}{2}+\epsilon}} \\ &\lesssim \|\nabla|^{\tilde{\epsilon}} A\|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|\phi_1\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}, \end{aligned}$$

where we used  $r - \frac{1}{2} \leq s \leq l$ .

**Proof of (30):** Assume first  $s \leq 1$ . By Sobolev's multiplication rule (17) and  $l \geq s > \frac{1}{2}$  we obtain

$$\begin{aligned} \|A_1 A_2 \phi\|_{L_t^2 H_x^{s-1}} &\lesssim \|A_1 A_2\|_{L_t^2 H_x^{0+}} \|\phi\|_{L_t^\infty H_x^s} \lesssim \|A_1\|_{L_t^4 H_x^{0+,4}} \|A_2\|_{L_t^4 H_x^{0+,4}} \|\phi\|_{L_t^\infty H_x^s} \\ &\lesssim \| |\nabla|^\epsilon A_1 \|_{L_t^4 H_x^{\frac{1}{2}-\epsilon+}} \| |\nabla|^\epsilon A_1 \|_{L_t^4 H_x^{\frac{1}{2}-\epsilon+}} \|\phi\|_{L_t^\infty H_x^s} \\ &\lesssim \| |\nabla|^\epsilon A_1 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}} \| |\nabla|^\epsilon A_2 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}. \end{aligned}$$

For  $s > 1$  the Sobolev multiplication law implies :

$$\|A_1 A_2 \phi\|_{L_t^2 H_x^{s-1}} \lesssim \|A_1 A_2\|_{L_t^3 H_x^{s-1}} \|\phi\|_{L_t^6 H_x^s}.$$

Now using  $l \geq s > 1$  we obtain

$$\begin{aligned} \|A_1 A_2\|_{L_t^3 H_x^{s-1}} &\lesssim \|\langle \nabla \rangle^{s-1} A_1 A_2\|_{L_t^3 L_x^2} + \|A_1 \langle \nabla \rangle^{s-1} A_2\|_{L_t^3 L_x^2} \\ &\lesssim \|\langle \nabla \rangle^{s-1} A_1\|_{L_t^6 L_x^4} \|A_2\|_{L_t^6 L_x^4} + \|A_1\|_{L_t^6 L_x^4} \|\langle \nabla \rangle^{s-1} A_2\|_{L_t^6 L_x^4} \\ &\lesssim \| |\nabla|^{\frac{1}{2}} A_1 \|_{L_t^6 H_x^{s-1}} \| |\nabla|^{\frac{1}{2}} A_2 \|_{L_t^6 L_x^2} + \| |\nabla|^{\frac{1}{2}} A_1 \|_{L_t^6 L_x^2} \| |\nabla|^{\frac{1}{2}} A_2 \|_{L_t^6 H_x^{s-1}} \\ &\lesssim \| |\nabla|^\epsilon A_1 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}} \| |\nabla|^\epsilon A_2 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}}, \end{aligned}$$

which gives the same bound for  $\|A_1 A_2 \phi\|_{L_t^2 H_x^{s-1}}$  as in the case  $s \leq 1$ .

Next, let us assume first that  $s \leq \frac{3}{4}$ . By Prop. 1.1 we obtain

$$\|A_1 A_2 \phi\|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \|A_1 A_2\|_{X_{|\tau|=|\xi|}^{-\frac{1}{4}+, 0}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}.$$

It remains to estimate  $\|A_1 A_2\|_{X_{|\tau|=|\xi|}^{-\frac{1}{4}+, 0}} = \|A_1 A_2\|_{L_t^2 H_x^{-\frac{1}{4}+}}$ . On the one hand we obtain for  $l > \frac{1}{2}$  and  $r > \frac{1}{4}$  by Sobolev

$$\begin{aligned} \|A_1 A_2\|_{L_t^2 H_x^{-\frac{1}{4}+}} &\lesssim \|A_1 A_2\|_{L_t^2 L_x^{\frac{8}{3}+}} \lesssim \|A_1\|_{L_t^4 L_x^4} \|A_2\|_{L_t^4 L_x^{\frac{8}{3}+}} \lesssim \|A_1\|_{L_t^4 \dot{H}_x^{\frac{1}{2}}} \|A_2\|_{L_t^4 H_x^{\frac{1}{4}+}} \\ &\lesssim \| |\nabla|^\epsilon A_1 \|_{L_t^4 H_x^{l-\epsilon}} \|A_2\|_{L_t^4 H_x^r} \lesssim \| |\nabla|^\epsilon A_1 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}. \end{aligned}$$

On the other hand we use Prop. 1.1 again and obtain for  $r > \frac{1}{4}$  :

$$\|A_1 A_2\|_{X_{|\tau|=|\xi|}^{-\frac{1}{4}+, 0}} \lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}+, \frac{1}{2}+}} \|A_2\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}+, \frac{1}{2}+}} \lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}},$$

which completes the proof in the case  $s \leq \frac{3}{4}$ .

Next, we assume  $s > \frac{3}{4}$ . By Prop. 1.1 we obtain

$$\|A_1 A_2 \phi\|_{X_{|\tau|=|\xi|}^{s-1, -\frac{1}{2}+2\epsilon}} \lesssim \|A_1 A_2\|_{X_{|\tau|=|\xi|}^{s-1, 0}} \|\phi\|_{X_{|\tau|=|\xi|}^{s, \frac{1}{2}+\epsilon}}.$$

It remains to estimate  $\|A_1 A_2\|_{X_{|\tau|=|\xi|}^{s-1, 0}} = \|A_1 A_2\|_{L_t^2 H_x^{s-1}}$ . On the one hand we apply Prop. 1.1 again, use  $r \geq s - \frac{1}{2}$  and obtain

$$\|A_1 A_2\|_{X_{|\tau|=|\xi|}^{s-1, 0}} \lesssim \|A_1\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}.$$

On the other hand, if  $\frac{3}{4} < s \leq 1$  we crudely estimate using  $l \geq s > \frac{3}{4}$  and  $r \geq s - \frac{1}{2} > \frac{1}{4}$  :

$$\begin{aligned} \|A_1 A_2\|_{L_t^2 H_x^{s-1}} &\leq \|A_1 A_2\|_{L_t^2 L_x^2} \leq \|A_1\|_{L_t^4 L_x^8} \|A_2\|_{L_t^4 L_x^{\frac{8}{3}}} \lesssim \| |\nabla|^{\frac{3}{4}} A_1 \|_{L_t^4 L_x^2} \|A_2\|_{L_t^4 H_x^{\frac{1}{4}}} \\ &\lesssim \| |\nabla|^\epsilon A_1 \|_{X_{\tau=0}^{l-\epsilon, \frac{1}{2}+\epsilon-}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}. \end{aligned}$$

If  $s > 1$  we use  $l \geq s > 1$  and  $r \geq s - \frac{1}{2} > \frac{1}{2}$  and obtain

$$\begin{aligned} \|A_1 A_2\|_{L_t^2 H_x^{s-1}} &\lesssim \|\langle \nabla \rangle^{s-1} A_1 A_2\|_{L_t^2 L_x^2} + \|A_1 \langle \nabla \rangle^{s-1} A_2\|_{L_t^2 L_x^2} \\ &\lesssim \|\langle \nabla \rangle^{s-1} A_1\|_{L_t^4 L_x^4} \|A_2\|_{L_t^4 L_x^4} + \|A_1\|_{L_t^4 L_x^4} \|\langle \nabla \rangle^{s-1} A_2\|_{L_t^4 L_x^4} \\ &\lesssim \| |\nabla|^{\frac{1}{2}} A_1 \|_{L_t^4 H_x^{s-1}} \| |\nabla|^{\frac{1}{2}} A_2 \|_{L_t^4 L_x^2} + \| |\nabla|^{\frac{1}{2}} A_1 \|_{L_t^4 L_x^2} \| |\nabla|^{\frac{1}{2}} A_2 \|_{L_t^4 H_x^{s-1}} \\ &\lesssim \| |\nabla|^{\tilde{\epsilon}} A_1 \|_{X_{\tau=0}^{l-\tilde{\epsilon}, \frac{1}{2}+\epsilon-}} \|A_2\|_{X_{|\tau|=|\xi|}^{r, \frac{3}{4}+\epsilon}}, \end{aligned}$$

which completes the proof in the case  $s > \frac{3}{4}$  and also the proof of (30) and part 1 of Theorem 1.1.

**Proof of part 2 of Theorem 1.1 :** The claimed regularity of the solution clearly holds. Let us now assume that the solution fulfills

$$\phi_{\pm}, A_{\pm}^{df} \in C^0([0, T], H^1(\mathbb{R}^2)), |\nabla|^{\tilde{\epsilon}} A^{cf} \in C^0([0, T], H^{1-\tilde{\epsilon}}(\mathbb{R}^2)).$$

We want to show that such a solution belongs to a space where uniqueness holds by part 1 of the theorem.

**Step 1:**  $A_{\pm}^{df} \in X_{\pm}^{1-, 1-}[0, T]$ .

We drop  $[0, T]$  from all the spaces in the sequel. Interpolation between Strichartz' estimate (18) and  $\|u\|_{L_{xt}^2} = \|u\|_{X_{\pm}^{0,0}}$  gives  $\|u\|_{L_t^{2+} L_x^{2+}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{0+, 0+}}$ , thus by duality

$\|u\|_{X_{|\tau|=|\xi|}^{0-, 0-}} \lesssim \|u\|_{L_t^{2-} L_x^{2-}}$ . Consequently,

$$\begin{aligned} \|\phi \overline{\nabla \phi}\|_{X_{\pm}^{0-, 0-}} &\lesssim \|\phi \overline{\nabla \phi}\|_{L_t^{2-} L_x^{2-}} \lesssim \|\phi\|_{L_t^{\infty} L_x^{\infty-}} \|\nabla \phi\|_{L_t^{\infty} L_x^2} T^{\frac{1}{2}+} \\ &\lesssim \|\phi\|_{L_t^{\infty} \dot{H}_x^{1-}} \|\phi\|_{L_t^{\infty} \dot{H}_x^1} T^{\frac{1}{2}+} < \infty \end{aligned}$$

Moreover

$$\begin{aligned} \|A|\phi|^2\|_{X_{\pm}^{0-, 0-}} &\lesssim \|A|\phi|^2\|_{L_t^{2-} L_x^{2-}} \lesssim \|A\|_{L_t^{\infty} L_x^{6-}} \|\phi\|_{L_t^{\infty} L_x^{6-}}^2 T^{\frac{1}{2}+} \\ &\lesssim \|A\|_{L_t^{\infty} \dot{H}_x^{\frac{2}{3}-}} \|\phi\|_{L_t^{\infty} \dot{H}_x^{\frac{2}{3}-}}^2 T^{\frac{1}{2}+} < \infty. \end{aligned}$$

By (12) we obtain the desired regularity.

**Step 2:**  $\phi_{\pm} \in X_{\pm}^{1-, 1-}[0, T]$ .

Using (13) this leads to the same estimates as in step 1.

**Step 3:**  $|\nabla|^{\tilde{\epsilon}} A^{cf} \in X_{\tau=0}^{\frac{3}{4}-\tilde{\epsilon}, \frac{1}{2}+}$ .

Using (11) and step 2 it suffices to show

$$\| |\nabla|^{-1+\tilde{\epsilon}} (\phi \overline{\partial_t \phi}) \|_{X_{\tau=0}^{\frac{3}{4}-\tilde{\epsilon}, -\frac{1}{2}+}} \lesssim \|\phi\|_{X_{|\tau|=|\xi|}^{1-, 1-}}^2.$$

Replacing as before  $|\nabla|^{-1+\tilde{\epsilon}}$  by  $\langle \nabla \rangle^{-1+\tilde{\epsilon}}$  this reduces to

$$\int_* \frac{\widehat{u_1}(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-} \langle |\tau_1| - |\xi_1| \rangle^{1-}} \frac{\widehat{u_2}(\xi_2, \tau_2) |\tau_2|}{\langle \xi_2 \rangle^{1-} \langle |\tau_2| - |\xi_2| \rangle^{1-}} \frac{\widehat{u_3}(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{4}-} \langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}.$$

Case 1:  $|\tau_2| \lesssim |\xi_2|$ . Using

$$|\tau_2| \lesssim \langle \xi_2 \rangle^{1-} \langle \xi_2 \rangle^{0+} \lesssim \langle \xi_2 \rangle^{1-} (\langle \xi_1 \rangle^{0+} + \langle \xi_3 \rangle^{0+})$$

we reduce to

$$\int_* \frac{\widehat{u_1}(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-} \langle |\tau_1| - |\xi_1| \rangle^{1-}} \frac{\widehat{u_2}(\xi_2, \tau_2)}{\langle |\tau_2| - |\xi_2| \rangle^{1-}} \frac{\widehat{u_3}(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{4}-} \langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

which holds by Sobolev.

Case 2:  $|\tau_2| \gg |\xi_2|$  and  $|\xi_1| \gtrsim |\tau_1|$ . In this case we have

$$|\tau_2| \lesssim ||\tau_2| - |\xi_2||^{1-} (\langle \tau_3 \rangle^{0+} + \langle \xi_1 \rangle^{0+}),$$

so that it suffices to show

$$\int_* \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{1-} \langle |\tau_1| - |\xi_1| \rangle^{1-}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^{1-}} \frac{\widehat{u}_3(\xi_3, \tau_3)}{\langle \xi_3 \rangle^{\frac{1}{4}} \langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

which also holds by Sobolev.

Case 3:  $|\tau_2| \gg |\xi_2|$  and  $|\tau_1| \gg |\xi_1|$ . In this case we obtain

$$|\tau_2| \lesssim \langle |\tau_2| - |\xi_2| \rangle^{1-} (\langle \tau_3 \rangle^{0+} + \langle |\tau_1| - |\xi_1| \rangle^{0+}),$$

which can be handled similarly as case 2.

The regularity obtained in steps 1-3 is more than sufficient to deduce the uniqueness by an application of part 1 of the theorem.  $\square$

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